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PLANE CONTACT PROBLEMS FOR A PHYSICALLY NON-LINEAR PRESTRESSED ELASTIC MEDIUM*

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The framework of the physically non-linear theory of elasticity is used to formulate and solve plane contact problems for a half-plane and a strip of finite depth, prestressed by the action of longitudinal, uniformly distributed forces applied at infinity. The cases of loss of stability in the medium induced by prestressing are studied and the influence of the prestressing mode on the magnitude of the contact pressures is investigated.

1. The resolving equations of the physically non-linear (geometrically linear) theory of elasticity can be written for the case of plane deformation, provided that there are no mass forces, in the form /1/

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \tag{1.1}$$

$$\varepsilon_x=\psi\sigma_x+(\varphi-\psi)\sigma, \quad \varepsilon_y=\psi\sigma_y+(\varphi-\psi)\sigma, \quad \varepsilon_z=0$$

$$\varepsilon_{xy}=\psi\tau_{xy}, \quad \varepsilon=\varphi\sigma, \quad \frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=2\frac{\partial^2\varepsilon_{xy}}{\partial x\,\partial y}$$

$$\sigma=\frac{\sigma_x+\sigma_y+\sigma_z}{3}, \quad \tau=\frac{1}{\sqrt{6}}[(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2+6\tau_{xy}^2]^{1/2}$$

Here $1/(2\psi)$ is the reduced shear modulus, $1/\varphi$ is the reduced volume deformation modulus and the functions $\psi=\psi(|\sigma|,\tau)$, $\varphi=\varphi(|\sigma|,\tau)$ are continuous, monotonic and positive with respect to both their arguments.

The initial stress state of the medium is given as follows:

$$\sigma_x^\circ=\pm p, \quad \sigma_y^\circ=\overset{\circ}{\tau}_{xy}=0, \quad \sigma^\circ=\pm\frac{\psi^\circ}{2\psi^\circ+\varphi^\circ}p \tag{1.2}$$

$$\tau^\circ=\mp\frac{[\varphi^{\circ 2}+\psi^{\circ 2}+\varphi^\circ\psi^\circ]^{1/2}}{2\psi^\circ+\varphi^\circ}p$$

Then, according to (1.1) the initial deformations have the form

$$\varepsilon_x^\circ=\pm\frac{\psi^\circ(\psi^\circ+2\varphi^\circ)}{2\psi^\circ+\varphi^\circ}p, \quad \varepsilon_y^\circ=\pm\frac{\psi^\circ(\varphi^\circ-\psi^\circ)}{2\psi^\circ+\varphi^\circ}p, \quad \overset{\circ}{\varepsilon}_{xy}=0$$

We further write $\sigma_x=\sigma_x^\circ+\sigma_x^*,\ldots$; $\varepsilon_x=\varepsilon_x^\circ+\varepsilon_x^*,\ldots$; $u=u^\circ+u^*,\ldots$; $\sigma=\sigma^\circ+\sigma^*$, $\tau=\tau^\circ+\tau^*$, $\psi=\psi^\circ-\psi^*$, $\varphi=\varphi^\circ-\varphi^*$ where $\psi^*=\psi_{|\sigma|}^{\circ\prime}|\sigma^*|+\psi_\tau^{\circ\prime}\tau^*$, $\varphi^*=\varphi_{|\sigma|}^{\circ\prime}|\sigma^*|+\varphi_\tau^{\circ\prime}\tau^*$, $\varphi^\circ=\varphi(|\sigma^\circ|,\tau^\circ)$, $\psi^\circ=\psi(|\sigma^\circ|,\tau^\circ)$, where the symbols with asterisks denote small perturbations in the basic stress, defromation and displacement fields. Linearizing Eqs.(1.1) with respect to these perturbations, we obtain /2/

$$\frac{\partial \sigma_x^*}{\partial x}+\frac{\partial \tau_{xy}^*}{\partial y}=0, \quad \frac{\partial \tau_{xy}^*}{\partial x}+\frac{\partial \sigma_y^*}{\partial y}=0 \tag{1.3}$$

$$\varepsilon_x^*=\frac{\psi^\circ}{n+2}(L_1\sigma_x^*+N_2\sigma_y^*), \quad \varepsilon_y^*=\frac{\psi^\circ}{n+2}(N_1\sigma_x^*+L_2\sigma_y^*)$$

$$\varepsilon_{xy}^*=\psi^\circ\tau_{xy}^*$$

$$L_k=1+2n+\lambda_k(m_1,m_2)\Sigma_k+\lambda_k(l_1,l_2)T_k$$

$$N_k=n-1+\lambda_{3-k}(m_1,m_2)\Sigma_k+\lambda_{3-k}(l_1,l_2)T_k, \quad k=1,2$$

$$\lambda_1(m_1,m_2)=[2(n^2+n+1)m_1+3m_2](n+2)^{-1}$$

$$\lambda_2(m_1,m_2)=[(n^2-2n-2)m_1+3m_2](n+2)^{-1}$$

$$\Sigma_1=\Delta^{-1}\left[2\sqrt{1+n+n^2}-\frac{1+2n}{n+2}(nl_1-l_2)\right]$$

$$T_1=\Delta^{-1}\left[-2(1+n+n^2)+\frac{1+2n}{n+2}(nm_1-m_2)\right]$$

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$$\Sigma_2=\Delta^{-1}\left[2\sqrt{1+n+n^2}-\frac{n-1}{n+2}(nl_1-l_2)\right]$$
$$T_2=\Delta^{-1}\left[-(n^2-2n-2)+\frac{n-1}{n+2}(nm_1-m_2)\right]$$
$$\Delta=2(n+2)\sqrt{1+n+n^2}-$$
$$-[3n(nl_1-l_2)+2\sqrt{1+n+n^2}(nm_1-m_2)](n+2)^{-1}$$
$$n=\frac{\varphi^\circ}{\psi^\circ},\quad m_1=\frac{\psi^{\circ\prime}_{|\sigma|}}{\psi^\circ},\quad m_2=\frac{\varphi^{\circ\prime}_{|\sigma|}}{\psi^\circ},\quad l_1=\frac{\psi_\tau^{\circ\prime}}{\psi^\circ},\quad l_2=\frac{\varphi_\tau^{\circ\prime}}{\psi^\circ}$$

Here we have introduced the dimensionless quantities n, m_1, m_2, l_1, l_2 characterizing the mechanical properties of the material of the medium in question, which take into account the prestressed mode. We also have

$$\frac{\partial^2\varepsilon_x^*}{\partial y^2}+\frac{\partial^2\varepsilon_y^*}{\partial x^2}=2\frac{\partial^2\varepsilon_{xy}^*}{\partial x\,\partial y} \tag{1.4}$$
$$\varepsilon_x^*=\frac{\partial u^*}{\partial x},\quad \varepsilon_y^*=\frac{\partial v^*}{\partial y},\quad \varepsilon_{xy}^*=\frac{1}{2}\left(\frac{\partial u^*}{\partial y}+\frac{\partial v^*}{\partial x}\right)$$

Let us introduce the Airy stress function

$$\sigma_x^*=\frac{\partial^2\Phi}{\partial y^2},\quad \sigma_y^*=\frac{\partial^2\Phi}{\partial x^2},\quad \tau_{xy}^*=-\frac{\partial^2\Phi}{\partial x\,\partial y} \tag{1.5}$$

Then Eq.(1.4) of the compatibility of deformations ε_x^*, ε_y^*, ε_{xy}^* can be written, using Eqs.(1.5), in the form

$$\frac{\partial^4}{\partial y^4}\Phi(x,y)+2A\frac{\partial^4}{\partial x^2\,\partial y^2}\Phi(x,y)+B\frac{\partial^4}{\partial x^4}\Phi(x,y)=0 \tag{1.6}$$
$$A=\frac{N_1+N_2+2(n+2)}{2L_1},\quad B=\frac{L_2}{L_1}$$

The conditions that Eq.(1.6) be elliptic have the form

$$\text{a) } A^2-B<0,\ \text{b) } A>0,\ B>0,\ A^2\geqslant B \tag{1.7}$$

Let us write these conditions of ellipticity, as an example, for some special cases of the functions $\psi(|\sigma|,\tau)$, $\varphi=(|\sigma|,\tau)$.

1° $\psi=\psi(\tau)$, $\varphi=0$ and we have $n=m_1=m_2=l_2=0$, $l_1=l\neq 0$. Then the equation is elliptic if a) $l<0$, b) $0\leqslant l<2$.

2° $\psi=\psi(|\sigma|\tau)$, $\varphi=0$ and we have $n=m_2=l_2=0$, $l_1=l\neq 0$, $m_1=m\neq 0$. Eq.(1.6) is elliptic if

$$\text{a) } l<-m^2/8\ (|m|<\infty),\ \text{b) }-m^2/8\leqslant l<2-|m|\ (|m|<4) \tag{1.8}$$

The shaded area in Fig.1 shows the domain of ellipticity for this case. It should be noted, noted, that if $\psi(|\sigma|,\tau)$ is a homogeneous function of its arguments, then the elliptic character of the equation does not depend on the initial stress of the medium and is determined by the structure of the function ψ.

3° $\psi=\psi(\tau)$, $\varphi=\text{const}$, and we have $m_1=m_2=l_2=0$, $l_1=l\neq 0$, $n\neq 0$. Eq.(1.6) will be elliptic, provided that

$$\begin{aligned} &\text{a) } l<0,\ l>M_1\ (n>-1/2);\ 0<l<M_1\ (n<-1/2)\\ &\text{b) } l<M_2(n<-2);\ M_1\leqslant l<M_2,\ l>M_3\ (-2\leqslant n\leqslant -1.22)\\ &M_3<l<0\ (-1.22<n<-1/2);\ 0<l<M_3\ (n>-1/2)\\ &M_2/(1+2n)<l<M_1\ (n\geqslant 2.73)\end{aligned} \tag{1.9}$$

$$M=2(n+2)^2\sqrt{1+n+n^2},\quad M_1=M/(3n^2)$$
$$M_2=\frac{M(1+2n)}{3n^2(1+2n)+(n^2-2n-2)^2},\quad M_3=\frac{M(1+2n)}{3n^2(1+2n)+4(1+n+n^2)^2}$$

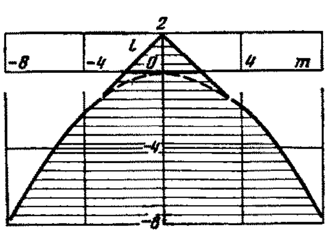

Fig.1

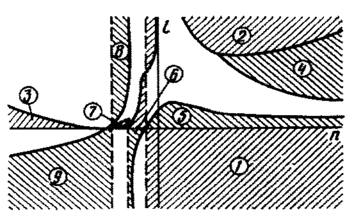

Fig.2

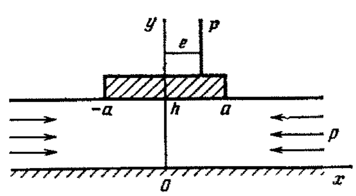

Fig.3

From (1.9) it follows that the critical values of m and l satisfy the equations

$$l = 0,\ l = M_1\ (-1.22 \leqslant n \leqslant 2.73);\ l = M_2,\ l = M_3\ (|n| < \infty)$$

The shaded areas in Fig.2 show the domains in which (1.6) is elliptic, corresponding to the case in question.

From the mechanical point of view, the cases in which (1.6) ceases to be elliptic can be treated as the cases in which the medium loses its internal stability as a result of the pre-stressing /3/.

2. Let us now consider the contact problem. Let a strip occupying the region $0 \leqslant y \leqslant h$, made of material obeying the relations (1.3), lie without friction on a rigid support. A rigid rectangular stamp (Fig.3) is impressed into the boundary of the strip $y = h$ by a force P whose eccentricity of application is e. We shall assume that the frictional forces in the area of contact between the stamp and the strip are small and can be neglected, and the width $2a$ of the area of contact is independent of the magnitude of the force applied. We will regard the action of the stamp on the strip as a small perturbation in the basic stress field (1.2).

Let us write the boundary conditions for this contact problem

$$\begin{gathered}\tau_{xy}^*(x, 0) = 0,\ v^*(x, 0) = 0,\ \tau_{xy}^*(x, h) = 0\ (|x| < \infty) \\ v^*(x, h) = -(\delta + \alpha x - f(x))\ \ (|x| \leqslant a),\quad \sigma_y^*(x, h) = 0\ \ (|x| > a)\end{gathered} \tag{2.1}$$

We must supplement these conditions with the demand that the stresses (additional with respect to the initial stress field) vanish at infinity. Here δ is the translational displacement of the stamp along the axis y, α is the angle of rotation of the stamp about the axis z, $f(x)$ is a function describing the form of the stamp foundation.

In order to construct the solution of the boundary value problem formulated above, we will apply the integral Fourier transform in the variable x to Eq.(1.6) in the region where it is elliptic, i.e. we shall seek the Airy function in the form

$$\Phi(x, y) = \int_{-\infty}^{\infty} \overline{\Phi}(\omega, y)\, e^{i\omega x}\, d\omega \tag{2.2}$$

Here Eq.(1.6) leads to an ordinary differential equation for the inverse Fourier transform $\overline{\Phi}(\omega, y)$ whose solution is given by the formula

$$\overline{\Phi}(\omega, y) = C_1^+(\omega) \operatorname{sh} \omega\varkappa_+ y + C_2^+(\omega) \operatorname{ch} \omega\varkappa_+ y + C_1^-(\omega) \operatorname{sh} \omega\varkappa_- y + C_2^-(\omega) \operatorname{ch} \omega\varkappa_- y \tag{2.3}$$

$$\text{a)}\ A^2 < B,\quad \varkappa_\pm = c_+ \pm ic_-,\quad c_\pm = \left(\frac{A \pm B^{1/2}}{2}\right)^{1/2}$$

$$\text{b)}\ A > 0,\ B > 0,\ A^2 \geqslant B,\ \varkappa_\pm = [A \pm (A^2 - B)^{1/2}]^{1/2}$$

We further introduce the function of contact pressure distribution

$$\sigma_y^*(x, h) = -q(x),\quad |x| \leqslant a \tag{2.4}$$

and, assuming for the time being, that it is known, we apply the integral Fourier transform (2.2) to the first three boundary conditions from (2.1), and to (2.4). Next, having satisfied the transformed boundary conditions with help of Eqs.(2.3), we obtain (the additional stresses vanish as $|x| \to \infty$ by virtue of the properties of Fourier integrals)

$$\begin{gathered}C_1^\pm(\omega) = 0,\quad C_2^\pm(\omega) = \mp Q(\omega)\, \omega^{-2} \varkappa_\mp \operatorname{sh} \omega\varkappa_\mp h \times \\ [\varkappa_+ \operatorname{sh} \omega\varkappa_+ h \operatorname{ch} \omega\varkappa_- h - \varkappa_- \operatorname{sh} \omega\varkappa_- h \operatorname{ch} \omega\varkappa_+ h]^{-1} \\ Q(\omega) = \frac{1}{2\pi} \int_{-a}^{a} q(x)\, e^{-i\omega x}\, dx\end{gathered} \tag{2.5}$$

(where $(Q(\omega)$ is the inverse Fourier transform of the contact pressure distribution function). Using the results obtained we can write the following expression for determining the settling under the stamp

$$v^*(x, h) = -\frac{a}{\theta} \int_{-1}^{1} q(\xi)\, K\left(\frac{x - \xi}{\lambda}\right) d\xi \tag{2.6}$$

Further, by satisfying the last boundary condition of (2.1), we reduce the contact problem in question to an integral equation of the first kind in $q(x)$

$$\int_{-1}^{1} q(\xi)\, K\left(\frac{x - \xi}{\lambda}\right) d\xi = \pi\theta(\delta + \alpha x - f(x)),\quad |x| \leqslant 1 \tag{2.7}$$

$$K\left(\frac{x-\xi}{\lambda}\right)=\int_0^\infty \frac{L(\omega)}{\omega}\cos\omega\frac{x-\xi}{\lambda}\,d\omega$$

$$L(\omega)=c_-\frac{\operatorname{ch}2c_+\omega-\cos 2c_-\omega}{c_-\operatorname{sh}2c_+\omega+c_+\sin 2c_-\omega},\quad \theta=\frac{n+2}{L_2}\quad\frac{c_+^2+c_-^2}{2c_+}$$

$$L(\omega)=\frac{\varkappa_+-\varkappa_-}{\varkappa_+\operatorname{cth}\omega\varkappa_--\varkappa_-\operatorname{cth}\omega\varkappa_+},\quad \theta=\frac{n+2}{L_2}\quad\frac{\varkappa_+\varkappa_-}{\varkappa_++\varkappa_-}$$

In relations (2.6), (2.7) $\lambda=h/a$ is a dimensionless parameter characterizing the strip thickness, and x, ξ, ω are dimensionless variables. It should be noted that as $\lambda\to\infty$, the integral equation for the prestressed, physically non-linear strip (2.7) will be transformed into an integral equation of the corresponding contact problem for a prestressed, physically non-linear elastic half-plane

$$\int_{-1}^{1} q(\xi)[-\ln|\xi-x|+C]\,d\xi=\pi\theta(\delta+\alpha x-f(x)),\quad |x|\leqslant 1 \tag{2.8}$$

Here the constant C is infinite. When the strip is very thick, the contact problem can /4/ also be reduced to integral Eq.(2.8), but the value of the constant C will be finite. The parameter θ represents an important characteristic of the problem for a half-plane, and in this case we can call this parameter the contact rigidity (CR).

Below we give the results of an investigation of the CR for some special cases of the function φ, ψ.

1° $\psi=\psi(\tau)$, $\varphi=0$. We have $\theta=(1-l/2)^{-1/2}$. In the domain of ellipticity $(-\infty<l<0,\ 0\leqslant l<2)$ θ takes real positive values and never vanishes. The dashed line in Fig.1 shows the dependence of CR on the degree m of prestressing of the medium. The cases when CR becomes infinite can be treated as cases of the loss of surface deformability of the medium caused by prestressing. The loss of surface deformability and the loss of internal stability occur simultaneously during the passage through the point $l=2$.

2° $\psi=\psi(|\sigma|,\tau)$, $\varphi=0$. We have

a) $\theta=2(2/\mu)^{1/2}$, $\mu=(2-m-l)[2+l+((2-l)^2-m^2)^{1/2}]$

b) $\theta=4/\mu$, $\mu=(2-m-l)^{1/2}[2+l+(m^2+8l)^{1/2}+(2+l-(m^2+8l)^{1/2})^{1/2}]$

As before, θ is real and positive in the region of variation of the parameters m and l (1.8) where Eq.(1.6) is elliptic. Outside this region θ is complex. On the segment of the contour on which $l=2+m\ (-4<m<0)$ θ is positive, and on the rest of the contour $l=2-m\ (0<m<4)$, $l=-m^2/8\ (m>4)$ it becomes infinite, i.e. we have the loss of surface deformability of the medium. Here, as in 1°, a loss of surface deformability and a loss of integral stability occur simultaneously.

3° $\psi=\psi(\tau)$, $\varphi=\text{const}$. We have $\theta=N(n+2)/\mu$ and

a) $\mu=S^{1/2}(S_4+(SS_5)^{1/2})^{1/2}$

b) $\mu=\chi(\chi S)^{1/2}[(\chi S_4+(n+2)^2((1+2n)lN)^{1/2})^{1/2}+(\chi S_4-(n+2)^2((1+2n)lN)^{1/2})^{1/2}]$

Here

$$N=M-3n^2l,\quad S=(1+2n)M-[3n^2(1+2n)+(n^2-2n-2)]\,l$$
$$S_4=(1+2n)M-[3n^2(1+2n)-2(1+n+n^2)(n^2-2n-2)]\,l$$
$$S_5=(1+2n)M-[3n^2(1+2n)+4(1+n+n^2)^2]\,l$$
$$\chi=\begin{cases}-1 & \text{in regions } 4,6,8 \quad \text{(Fig.2)}\\ +1 & \text{in regions } 5,7,9 \quad \text{(Fig.2)}\end{cases}$$

Unlike Examples 1° and 2°, in the region of variation of the parameters n and l of the form (1.9), where (1.6) is elliptic, θ can be positive as well as negative. The regions in Fig.2 marked *4, 6, 7, 9*, must be excluded from the discussion, since the negative value of CR contradicts the physical meaning of the problem. It is interesting to note that the regions in question are bounded by contours, passage through which is accompanied either by a loss of surface deformability of the medium $(\theta=\infty)$, or by a loss of surface stability of the medium $(\theta=0)$ occurring simultaneously with a loss of internal stability of the medium caused by prestressing. The CR becomes infinite during the passage across the contours $l=0\ (n\leqslant-1/2)$, $l=M_2\ (n\in(-\infty,-1{,}22)\setminus 2)$, $n=-1/2\ (l\in(-\infty,\infty)\setminus 5.19)$, and becomes zero when $l=M_1$ for any fixed value of the parameter n and for $n=-2$ when l is arbitrary. The dependence of CR on the mechanical properties of the material and prestressing is shown in Fig.4 by the solid lines.

3. Let us consider in more detail the integral equation of the contact problem for a prestressed, physically non-linear elastic strip (2.3). Formally, the equation is completely the same as the integral equation of the contact problem of a linearly elastic strip, and differs from it only in the form of the function $L(\omega)$ and the value of the dimensionless parameter θ. Analysis of expression (2.7) showed that the function $L(\omega)$ has all the characteristic

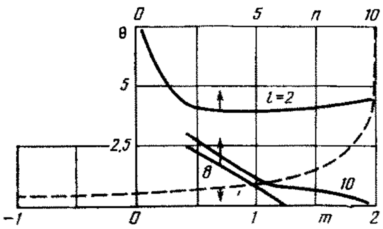

Fig.4

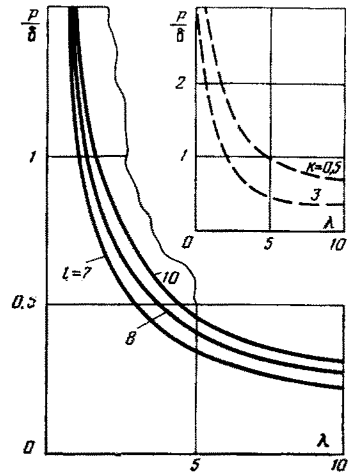

Fig.5

properties listed in /3/, namely 1) in the plane of the complex variable $z = \omega + i\omega_1$ the function $L(z)z^{-1}$ is even and meromorphic, and is real and regular when $\omega_1 = 0$; 2) $\lim L(z)z^{-1} = \varkappa_+\varkappa_-/(\varkappa_+ + \varkappa_-) > 0$ as $z \to 0$; 3) the following estimate holds on the real axis as $|\omega| \to \infty$:

$$L(\omega) = |\omega|^{-1}[1 + O(e^{-\nu|\omega|})] \qquad \nu = 2(\varkappa_+ + \varkappa_-) > 0$$

From this it follows that the asymptotic methods of "large" and "small" λ can be used to solve integral Eq.(2.7).

Figure 5 shows the results of an investigation of Eq.(2.7) for a flat stamp $(f(x) = 0)$ using the above methods. The results are shown in the form of the dependence of the magnitude of the impressing force, for a fixed depth of impression, on the relative layer thickness for materials with varying mechanical properties and conditions of prestressing (the solid lines refer to the case $\psi = \psi(|\sigma|, \tau), \varphi = 0, n = 3$, and the dashed lines to the case $\psi = \psi(|\sigma|, \tau), \varphi = k\psi, l = 1$).

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